

ARITHMETIC GENUS OF INTEGRAL SPACE CURVES

HAO SUN

ABSTRACT. We give an estimation for the arithmetic genus of an integral space curve, which are not contained in a surface of degree $k-1$. Our main technique is the Bogomolov-Gieseker type inequality for \mathbb{P}^3 proved by Macrì.

1. INTRODUCTION

A classical problem, which goes back to Halphen [6], is to determine, for given integers d and k , the maximal genus $G(d, k)$ of a smooth projective space curve of degree d not contained in a surface of degree $< k$. This problem is actually very natural, and has been investigated by many people (see [5, 7, 8, 9]).

In this paper, we consider the same problem for an integral space curve. Our main result is:

Theorem 1.1. *Let C be an integral complex projective curve in \mathbb{P}^3 of degree d . Let $p_a(C)$ be its arithmetic genus. If C is not contained in a surface of degree $< k$. Then*

$$p_a(C) \leq \begin{cases} \frac{2d^2}{3k} + \frac{1}{3}d(k-6) + 1, & \text{if } k^2 < d, \\ d(\sqrt{d}-2) + 1, & \text{if } k^2 \geq d. \end{cases}$$

The idea of the proof of Theorem 1.1 is to establish the tilt-stability of \mathcal{I}_C via computing its walls, then the Bogomolov-Gieseker type inequality for \mathbb{P}^3 proved by Macrì [12] implies Theorem 1.1. This Bogomolov-Gieseker type inequality naturally appears in the construction of Bridgeland stability conditions on threefolds (cf. [4, 3, 2]). There are also some other interesting applications of the Bogomolov-Gieseker type inequality in [1] and [13].

Our tilt-stability of \mathcal{I}_C can also give a version of the Halphen Speciality Theorem:

Theorem 1.2. *Let $C \subset \mathbb{P}^3$ be an integral complex projective degree d curve not contained in any surface of degree $< k$. Then $h^2(\mathcal{I}_C(l)) = h^1(\mathcal{O}_C(l)) = 0$, if $l > \frac{2d}{k} - 4$ when $k^2 < d$, or $l > 2\sqrt{d} - 4$ when $k^2 \geq d$.*

Our paper is organized as follows. In Section 2, we review basic properties of tilt-stability, the conjectural inequality proposed in [3, 2] and variants of the classical Bogomolov-Gieseker inequality satisfied by tilt-stable objects. Then in Section 3 the tilt-stability of \mathcal{I}_C has been established via computing its walls. Finally, we show the proof of Theorem 1.1 and 1.2 in Section 4.

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Notation. In this paper, we will always denote by C an integral projective curve in the three dimensional complex projective space \mathbb{P}^3 and by \mathcal{I}_C its ideal sheaf in \mathbb{P}^3 . We let $p_a(C) := h^1(C, \mathcal{O}_C)$ be the arithmetic genus of C . By X we denote a complex smooth projective threefold and by $D^b(X)$ its bounded derived category of coherent sheaves.

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2. PRELIMINARIES

In this section, we review the notion of tilt-stability for threefolds introduced in [3, 2]. Then we recall the Bogomolov-Gieseker type inequality for tilt-stable complexes proposed there.

Let X be a smooth projective threefold over \mathbb{C} , and let H be an ample divisor on X . Let $\alpha > 0$ and β be two real numbers. We write $\text{ch}^\beta(E) = e^{-\beta H} \text{ch}(E)$ denotes the Chern character twisted by βH . More explicitly, we have

$$\begin{aligned} \text{ch}_0^\beta &= \text{ch}_0 = \text{rank} & \text{ch}_2^\beta &= \text{ch}_2 - \beta H \text{ch}_1 + \frac{\beta^2}{2} H^2 \text{ch}_0 \\ \text{ch}_1^\beta &= \text{ch}_1 - \beta H \text{ch}_0 & \text{ch}_3^\beta &= \text{ch}_3 - \beta H \text{ch}_2 + \frac{\beta^2}{2} H^2 \text{ch}_1 - \frac{\beta^3}{6} H^3 \text{ch}_0. \end{aligned}$$

Slope-stability. We define the slope μ_β of a coherent sheaf $E \in \text{Coh}(X)$ by

$$\mu_\beta(E) = \begin{cases} +\infty, & \text{if } \text{ch}_0^\beta(E) = 0, \\ \frac{H^2 \text{ch}_2^\beta(E)}{H^3 \text{ch}_0^\beta(E)}, & \text{otherwise,} \end{cases}$$

Definition 2.1. A coherent sheaf E on X is slope-(semi)stable (or μ_β -(semi)stable) if, for all non-zero subsheaves $F \hookrightarrow E$, we have

$$\mu_\beta(F) < (\leq) \mu_\beta(E/F).$$

Harder-Narasimhan filtrations (HN-filtrations, for short) with respect to slope-stability exist in $\text{Coh}(X)$: given a non-zero sheaf $E \in \text{Coh}(X)$, there is a filtration

$$0 = E_0 \subset E_1 \subset \cdots \subset E_n = E$$

such that: $G_i := E_i/E_{i-1}$ is slope-semistable, and $\mu_\beta(G_1) > \cdots > \mu_\beta(G_n)$. We set $\mu_\beta^+(E) := \mu_\beta(G_1)$ and $\mu_\beta^-(E) := \mu_\beta(G_n)$.

Tilt-stability. There exists a *torsion pair* $(\mathcal{T}_\beta, \mathcal{F}_\beta)$ in $\text{Coh}(X)$ defined as follows:

$$\begin{aligned} \mathcal{T}_\beta &= \{E \in \text{Coh}(X) : \mu_\beta^-(E) > 0\} \\ \mathcal{F}_\beta &= \{E \in \text{Coh}(X) : \mu_\beta^+(E) \leq 0\} \end{aligned}$$

Equivalently, \mathcal{T}_β and \mathcal{F}_β are the extension-closed subcategories of $\text{Coh}(X)$ generated by slope-stable sheaves of positive and non-positive slope, respectively.

Definition 2.2. We let $\text{Coh}^\beta(X) \subset D^b(X)$ be the extension-closure

$$\text{Coh}^\beta(X) = \langle \mathcal{T}_\beta, \mathcal{F}_\beta[1] \rangle.$$

By the general theory of torsion pairs and tilting [10], $\text{Coh}^\beta(X)$ is the heart of a bounded t-structure on $\text{D}^b(X)$; in particular, it is an abelian category.

Now we can define the following slope function on $\text{Coh}^\beta(X)$: for an object $E \in \text{Coh}^\beta(X)$, we set

$$\nu_{\alpha,\beta}(E) = \begin{cases} +\infty, & \text{if } H^2 \text{ch}_1^\beta(E) = 0, \\ \frac{H \text{ch}_2^\beta(E) - \frac{1}{2}\alpha^2 H^3 \text{ch}_0^\beta(E)}{H^2 \text{ch}_1^\beta(E)}, & \text{otherwise.} \end{cases}$$

Definition 2.3. An object $E \in \text{Coh}^\beta(X)$ is *tilt-(semi)stable* (or $\nu_{\alpha,\beta}$ -(semi)stable) if, for all non-trivial subobjects $F \hookrightarrow E$, we have

$$\nu_{\alpha,\beta}(F) < (\leq) \nu_{\alpha,\beta}(E/F).$$

Lemma 3.2.4 in [3] shows that the Harder-Narasimhan property holds with respect to $\nu_{\alpha,\beta}$ -stability, i.e., for any $\mathcal{E} \in \text{Coh}^\beta(X)$ there is a filtration in $\text{Coh}^\beta(X)$

$$0 = \mathcal{E}_0 \subset \mathcal{E}_1 \subset \cdots \subset \mathcal{E}_n = \mathcal{E}$$

such that: $\mathcal{F}_i := \mathcal{E}_i / \mathcal{E}_{i-1}$ is $\nu_{\alpha,\beta}$ -semistable with $\nu_{\alpha,\beta}(\mathcal{F}_1) > \cdots > \nu_{\alpha,\beta}(\mathcal{F}_n)$.

Definition 2.4. In the above filtration, we call \mathcal{E}_1 the $\nu_{\alpha,\beta}$ -maximal subobject of $\mathcal{E} \in \text{Coh}^\beta(X)$. If \mathcal{E} is $\nu_{\alpha,\beta}$ -semistable, we say \mathcal{E} itself to be its $\nu_{\alpha,\beta}$ -maximal subobject.

Bogomolov-Gieseker type inequality. We now recall the Bogomolov-Gieseker type inequality for tilt-stable complexes proposed in [3, 2].

Definition 2.5. We define the generalized discriminant

$$\overline{\Delta}_H^\beta := (H^2 \text{ch}_1^\beta)^2 - 2H^3 \text{ch}_0^\beta \cdot (H \text{ch}_2^\beta).$$

A short calculation shows $\overline{\Delta}_H^\beta = (H^2 \text{ch}_1)^2 - 2H^3 \text{ch}_0 \cdot (H \text{ch}_2)$. Hence the generalized discriminant is independent of β .

Theorem 2.6 ([3, Theorem 7.3.1]). *Assume $E \in \text{Coh}^\beta(X)$ is $\nu_{\alpha,\beta}$ -semistable. Then*

$$(2.1) \quad \overline{\Delta}_H^\beta(E) \geq 0.$$

Conjecture 2.7 ([2, Conjecture 4.1]). *Assume $E \in \text{Coh}^\beta(X)$ is $\nu_{\alpha,\beta}$ -semistable. Then*

$$(2.2) \quad \alpha^2 \overline{\Delta}_H^\beta(E) + 4 \left(H \text{ch}_2^\beta(E) \right)^2 - 6H^2 \text{ch}_1^\beta(E) \text{ch}_3^\beta(E) \geq 0.$$

Such inequality was proved by Macrì [12] in the case of the projective space \mathbb{P}^3 :

Theorem 2.8. *The inequality (2.2) holds for $\nu_{\alpha,\beta}$ -semistable objects in $\text{D}^b(\mathbb{P}^3)$.*

3. TILT-STABILITY OF IDEAL SHEAVES OF SPACE CURVES

In this section, we establish the tilt-stability of ideal sheaves of spaces curves via computing their walls. Then from Theorem (2.8), we can deduce a Castelnuovo type inequality for integral curves in \mathbb{P}^3 .

Throughout this section, let C be an integral projective curve in \mathbb{P}^3 of degree d not contained in a surface of degree $< k$, and let \mathcal{I}_C be the ideal sheaf of C in

\mathbb{P}^3 . We keep the same notation as that in the previous section for $X = \mathbb{P}^3$ and $H =$ a plane of \mathbb{P}^3 . To simplify, we directly identify $H^{3-i} \text{ch}_i^\beta(E) = \text{ch}_i^\beta(E)$ for $E \in D^b(\mathbb{P}^3)$. The tilted slope becomes:

$$\nu_{\alpha,\beta} = \frac{\text{ch}_2^\beta - \frac{1}{2}\alpha^2 \text{ch}_0^\beta}{\text{ch}_1^\beta} = \frac{\text{ch}_2 - \beta \text{ch}_1 + \frac{1}{2}(\beta^2 - \alpha^2) \text{ch}_0}{\text{ch}_1 - \beta \text{ch}_0}.$$

The following lemma is a key observation for us to establish the tilt-stability of \mathcal{I}_C .

Lemma 3.1. *Let E be the $\nu_{\alpha,\beta}$ -maximal subobject of $\mathcal{I}_C \in \text{Coh}^\beta(\mathbb{P}^3)$ for some $(\alpha, \beta) \in \mathbb{R}_{>0} \times \mathbb{R}$. If $2\alpha^2 + \beta^2 \geq 4d$, then $\text{ch}_0(E) = 1$.*

Proof. By the long exact sequence of cohomology sheaves induced by the short exact sequence

$$0 \rightarrow E \rightarrow \mathcal{I}_C \rightarrow Q \rightarrow 0$$

in $\text{Coh}^\beta(\mathbb{P}^3)$, one sees that E is a torsion free sheaf with $\text{ch}_0(E) \geq 1$. If \mathcal{I}_C is $\nu_{\alpha,\beta}$ -semistable, then $E = \mathcal{I}_C$ by our definition. Hence $\text{ch}_0(E) = 1$.

Now we assume that \mathcal{I}_C is not $\nu_{\alpha,\beta}$ -semistable. One deduces

$$\nu_{\alpha,\beta}(E) = \frac{\text{ch}_2^\beta(E) - \frac{1}{2}\alpha^2 \text{ch}_0(E)}{\text{ch}_1^\beta(E)} > \nu_{\alpha,\beta}(\mathcal{I}_C) = \frac{\frac{1}{2}(\beta^2 - \alpha^2) - d}{-\beta},$$

i.e.,

$$(3.1) \quad \text{ch}_2^\beta(E) > \frac{\frac{1}{2}(\beta^2 - \alpha^2) - d}{-\beta} \text{ch}_1^\beta(E) + \frac{1}{2}\alpha^2 \text{ch}_0(E).$$

By Theorem 2.6, we obtain

$$(3.2) \quad \frac{(\text{ch}_1^\beta(E))^2}{2 \text{ch}_0(E)} \geq \text{ch}_2^\beta(E).$$

Combining (3.1) and (3.2), one sees that

$$\alpha^2 (\text{ch}_0(E))^2 + \frac{\beta^2 - \alpha^2 - 2d}{-\beta} \text{ch}_1^\beta(E) \text{ch}_0(E) - (\text{ch}_1^\beta(E))^2 < 0.$$

This implies

$$(3.3) \quad \text{ch}_0(E) < \left(\frac{\beta^2 - \alpha^2 - 2d}{\beta} + \sqrt{\left(\frac{\beta^2 - \alpha^2 - 2d}{\beta} \right)^2 + 4\alpha^2} \right) \frac{\text{ch}_1^\beta(E)}{2\alpha^2}.$$

Since E is a subobject of \mathcal{I}_C in $\text{Coh}^\beta(\mathbb{P}^3)$, by the definition of $\text{Coh}^\beta(\mathbb{P}^3)$, we deduce that

$$0 < \text{ch}_1^\beta(E) \leq \text{ch}_1^\beta(\mathcal{I}_C) = -\beta.$$

From (3.3), it follows that

$$(3.4) \quad \text{ch}_0(E) < \frac{(\alpha^2 - \beta^2 + 2d) + \sqrt{(\beta^2 - \alpha^2 - 2d)^2 + 4\alpha^2\beta^2}}{2\alpha^2}.$$

On the other hand, since $2\alpha^2 + \beta^2 \geq 4d$, a direct computation shows

$$\frac{(\alpha^2 - \beta^2 + 2d) + \sqrt{(\beta^2 - \alpha^2 - 2d)^2 + 4\alpha^2\beta^2}}{2\alpha^2} \leq 2.$$

Therefore, by (3.4), we conclude that $\text{ch}_0(E) < 2$, i.e., $\text{ch}_0(E) = 1$. \square

We now compute the walls of \mathcal{I}_C . See [11] for the surface case.

Lemma 3.2. *Let E be a subobject of \mathcal{I}_C in $\text{Coh}^\beta(\mathbb{P}^3)$ with*

$$(\text{ch}_0(E), \text{ch}_1(E), \text{ch}_2(E)) = (r, \theta, c).$$

Then $\nu_{\alpha, \beta}(E) \leq (<) \nu_{\alpha, \beta}(\mathcal{I}_C)$ if and only if

$$\frac{\theta}{2}(\alpha^2 + \beta^2) - (c + rd)\beta + \theta d \leq (<) 0.$$

Proof. Since E is a subobject of \mathcal{I}_C in $\text{Coh}^\beta(\mathbb{P}^3)$, one has

$$0 < \text{ch}_1^\beta(E) = \theta - r\beta \leq \text{ch}_1^\beta(\mathcal{I}_C) = -\beta,$$

i.e., $r\beta < \theta \leq (r-1)\beta \leq 0$.

Hence

$$\nu_{\alpha, \beta}(E) = \frac{\frac{r}{2}(\beta^2 - \alpha^2) - \beta\theta + c}{\theta - r\beta} \leq (<) \nu_{\alpha, \beta}(\mathcal{I}_C) = \frac{\frac{1}{2}(\beta^2 - \alpha^2) - d}{-\beta}$$

is equivalent to

$$-\beta \left(\frac{r}{2}(\beta^2 - \alpha^2) - \beta\theta + c \right) \leq (<) (\theta - r\beta) \left(\frac{1}{2}(\beta^2 - \alpha^2) - d \right),$$

i.e.,

$$\frac{\theta}{2}(\alpha^2 + \beta^2) - (c + rd)\beta + \theta d \leq (<) 0.$$

□

Proposition 3.3. *If $k^2 < d$, then \mathcal{I}_C is $\nu_{\alpha, \beta}$ -semistable for any $\alpha > 0$ and $\beta = -\frac{2d}{k}$.*

Proof. We let α_0 be an arbitrary positive real number, $\beta_0 = -\frac{2d}{k}$, and let E be the ν_{α_0, β_0} -maximal subobject of $\mathcal{I}_C \in \text{Coh}^{\beta_0}(\mathbb{P}^3)$.

Since $k^2 < d$, one sees that $2\alpha_0^2 + \beta_0^2 > \beta_0^2 > 4d$. Hence, by Lemma 3.1, one has $\text{ch}_0(E) = 1$, and E is subsheaf of \mathcal{I}_C . We can write $E = \mathcal{I}_W(-l)$, where $W \subset \mathbb{P}^3$ is a scheme of dimension ≤ 1 and $l \geq 0$. The Chern characters of $\mathcal{I}_W(-l)$ are

$$(\text{ch}_0(\mathcal{I}_W(-l)), \text{ch}_1(\mathcal{I}_W(-l)), \text{ch}_2(\mathcal{I}_W(-l))) = (1, -l, \frac{1}{2}l^2 + \text{ch}_2(\mathcal{I}_W)).$$

Since $\mathcal{I}_W(-l)$ is a subobject of \mathcal{I}_C in $\text{Coh}^{\beta_0}(\mathbb{P}^3)$, one deduce

$$0 < \text{ch}_1^{\beta_0}(\mathcal{I}_W(-l)) = -l - \beta_0 \leq \text{ch}_1^{\beta_0}(\mathcal{I}_C) = -\beta_0,$$

i.e.,

$$(3.5) \quad 0 \leq l < -\beta_0.$$

If $C \subseteq W$, then $\text{ch}_2(\mathcal{I}_W) \leq \text{ch}_2(\mathcal{I}_C) = -d$. Thus one sees that

$$\begin{aligned} \frac{-l}{2}(\alpha_0^2 + \beta_0^2) - \left(\frac{1}{2}l^2 + \text{ch}_2(\mathcal{I}_W) + d\right)\beta_0 - ld &\leq \frac{-l}{2}\beta_0^2 - \left(\frac{1}{2}l^2 - d + d\right)\beta_0 \\ &= \frac{-\beta_0 l}{2}(l + \beta_0) \\ &\leq 0. \end{aligned}$$

By Lemma 3.2, we conclude that $\nu_{\alpha_0, \beta_0}(\mathcal{I}_W(-l)) \leq \nu_{\alpha_0, \beta_0}(\mathcal{I}_C)$. Therefore the ν_{α_0, β_0} -maximal subobject of \mathcal{I}_C in $\text{Coh}^{\beta_0}(\mathbb{P}^3)$ is \mathcal{I}_C itself. Namely, \mathcal{I}_C is ν_{α_0, β_0} -semistable.

If $C \not\subseteq W$, then $\mathcal{I}_W(-l) \subset \mathcal{I}_C$ implies $\mathcal{O}_{\mathbb{P}^3}(-l) \subset \mathcal{I}_C$. Thus $l \geq k$. One deduces by (3.5) that

$$\begin{aligned}
 \frac{-l}{2}(\alpha_0^2 + \beta_0^2) - \left(\frac{1}{2}l^2 + \text{ch}_2(I_W) + d\right)\beta_0 - ld &< \frac{-l}{2}\beta_0^2 - \left(\frac{1}{2}l^2 + d\right)\beta_0 - ld \\
 &= \frac{-l}{2}\left(\beta_0^2 + \left(l + \frac{2d}{l}\right)\beta_0 + 2d\right) \\
 (3.6) \quad &= \frac{-l}{2}(\beta_0 + l)\left(\beta_0 + \frac{2d}{l}\right) \\
 &= \frac{-l}{2}(\beta_0 + l)\left(\frac{2d}{l} - \frac{2d}{k}\right) \\
 &\leq 0.
 \end{aligned}$$

From Lemma 3.2, it follows that \mathcal{I}_C is also ν_{α_0, β_0} -semistable in this case. \square

Proposition 3.4. *If $k^2 \geq d$, then \mathcal{I}_C is $\nu_{\alpha, \beta}$ -semistable for any $\alpha > 0$ and $\beta = -2\sqrt{d}$.*

Proof. The proof is almost the same as that of Proposition 3.3. We let α_0 be an arbitrary positive real number, $\beta_0 = -2\sqrt{d}$, and let E be the ν_{α_0, β_0} -maximal subobject of $\mathcal{I}_C \in \text{Coh}^{\beta_0}(\mathbb{P}^3)$.

By Lemma 3.1, the assumption $\beta_0 = -2\sqrt{d}$ makes sure that $\text{ch}_0(E) = 1$. We can still write $E = \mathcal{I}_W(-l)$ as in the proof of Proposition 3.3. When $C \subseteq W$, the same proof of Proposition 3.3 shows that \mathcal{I}_C is ν_{α_0, β_0} -semistable.

In the case of $C \not\subseteq W$, one sees that $l \geq k$. Thus it follows from (3.6) and (3.5) that

$$\begin{aligned}
 \frac{-l}{2}(\alpha_0^2 + \beta_0^2) - \left(\frac{1}{2}l^2 + \text{ch}_2(I_W) + d\right)\beta_0 - ld &< \frac{-l}{2}(\beta_0 + l)\left(\beta_0 + \frac{2d}{l}\right) \\
 &\leq \frac{-l}{2}(\beta_0 + l)\left(\frac{2d}{k} - 2\sqrt{d}\right).
 \end{aligned}$$

The assumption $k^2 \geq d$ guarantees that the left hand side of the above inequality is negative. Therefore we are done by Lemma 3.2. \square

4. THE PROOF OF THE MAIN THEOREMS

Now we can prove Theorem 1.1 and 1.2 easily.

Proof of Theorem 1.1. Since C is an integral curve, one sees that

$$\text{ch}_3^\beta(\mathcal{I}_C) = -\frac{1}{6}\beta^3 + d\beta + 2d - \chi(\mathcal{O}_C).$$

If \mathcal{I}_C is $\nu_{\alpha, \beta}$ -semistable, then from Theorem 2.8, it follows that

$$\begin{aligned}
 &\alpha^2 \overline{\Delta}_H^\beta(\mathcal{I}_C) + 4 \left(H \text{ch}_2^\beta(\mathcal{I}_C) \right)^2 - 6H^2 \text{ch}_1^\beta(\mathcal{I}_C) \text{ch}_3^\beta(\mathcal{I}_C) \\
 = & 2\alpha^2 d + 4d^2 + \beta^4 - 4\beta^2 d - 6(-\beta) \left(-\frac{1}{6}\beta^3 + d\beta + 2d - \chi(\mathcal{O}_C) \right) \\
 = & 2\alpha^2 d + 4d^2 + 2\beta^2 d + 6\beta(2d - \chi(\mathcal{O}_C)) \\
 \geq & 0,
 \end{aligned}$$

i.e.,

$$(4.1) \quad h^1(\mathcal{O}_C) - 1 = -\chi(\mathcal{O}_C) \leq \frac{2d^2 + (\alpha^2 + \beta^2)d}{3(-\beta)} - 2d.$$

By Proposition 3.3 and 3.4, one can substitute $(\alpha, \beta) = (0, -\frac{2d}{k})$ and $(\alpha, \beta) = (0, -2\sqrt{d})$ into (4.1) respectively to obtain our desired conclusion. \square

Proof of Theorem 1.2. The short exact sequence

$$0 \rightarrow \mathcal{I}_C(m) \rightarrow \mathcal{O}_{\mathbb{P}^3}(m) \rightarrow \mathcal{O}_C(m) \rightarrow 0$$

induces a long exact sequence

$$H^1(\mathcal{O}_{\mathbb{P}^3}(m)) \rightarrow H^1(\mathcal{O}_C(m)) \rightarrow H^2(\mathcal{I}_C(m)) \rightarrow H^2(\mathcal{O}_{\mathbb{P}^3}(m)).$$

Since $H^1(\mathcal{O}_{\mathbb{P}^3}(m)) = H^2(\mathcal{O}_{\mathbb{P}^3}(m)) = 0$, we deduce $h^2(\mathcal{I}_C(m)) = h^1(\mathcal{O}_C(m))$.

Now we assume that

Assumption 4.1. $m > \frac{2d}{k}$, $k^2 < d$ and $\beta_0 = -\frac{2d}{k}$.

One sees that

$$\text{ch}_1^{\beta_0}(\mathcal{O}_{\mathbb{P}^3}(-m)) = -m + \frac{2d}{k} < 0.$$

Thus $\mathcal{O}_{\mathbb{P}^3}(-m)[1] \in \text{Coh}^{\beta_0}(\mathbb{P}^3)$. It turns out that

$$\nu_{\alpha_0, \beta_0}(\mathcal{O}_{\mathbb{P}^3}(-m)[1]) = \frac{-\frac{1}{2}(m + \beta_0)^2 + \frac{1}{2}\alpha_0^2}{m + \beta_0} < \nu_{\alpha_0, \beta_0}(\mathcal{I}_C) = \frac{\frac{1}{2}(\beta_0^2 - \alpha_0^2) - d}{-\beta_0}$$

is equivalent to

$$-\beta_0 \left(-\frac{1}{2}(m + \beta_0)^2 + \frac{1}{2}\alpha_0^2 \right) < (m + \beta_0) \left(\frac{1}{2}(\beta_0^2 - \alpha_0^2) - d \right),$$

i.e.,

$$\alpha_0^2 + \beta_0^2 + (m + \frac{2d}{m})\beta_0 + 2d < 0.$$

Assumption 4.1 implies

$$\begin{aligned} \beta_0^2 + (m + \frac{2d}{m})\beta_0 + 2d &= (\beta_0 + m)(\beta_0 + \frac{2d}{m}) \\ &= (\beta_0 + m) \left(\frac{2d}{m} - \frac{2d}{k} \right) \\ &< (\beta_0 + m) \left(k - \frac{2d}{k} \right) \\ &< 0. \end{aligned}$$

Thus we can find an $\alpha_0 > 0$ such that $\nu_{\alpha_0, \beta_0}(\mathcal{O}_{\mathbb{P}^3}(-m)[1]) < \nu_{\alpha_0, \beta_0}(\mathcal{I}_C)$. On the other hand, by [3, Proposition 7.4.1] and Proposition 3.3, one deduces that $\mathcal{O}_{\mathbb{P}^3}(-m)[1]$ and \mathcal{I}_C are both ν_{α_0, β_0} -semistable. We conclude that

$$\text{Hom}_{\mathbb{D}^b(\mathbb{P}^3)}(\mathcal{I}_C, \mathcal{O}_{\mathbb{P}^3}(-m)[1]) = 0.$$

By the Serre duality theorem, one obtains $h^2(\mathcal{I}_C(m-4)) = 0$. Therefore we conclude that $h^2(\mathcal{I}_C(l)) = h^1(\mathcal{O}_C(l)) = 0$ if $l > \frac{2d}{k} - 4$ and $k^2 < d$.

Similarly, one can show $h^2(\mathcal{I}_C(l)) = h^1(\mathcal{O}_C(l)) = 0$ if $l > 2\sqrt{d} - 4$ and $k^2 \geq d$. \square

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DEPARTMENT OF MATHEMATICS, SHANGHAI NORMAL UNIVERSITY, SHANGHAI 200234, PEOPLE'S REPUBLIC OF CHINA

E-mail address: `hsun@shnu.edu.cn`